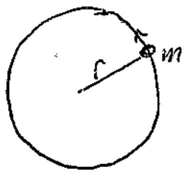
Rotation in two dimensions
(particle on a ring)

This is another problem for which we can solve exactly.



Let consider a particle of mass m freely moving in a circular path of radius r. Thus, it has only one degree of freedom, given by ϕ and the potential energy, V, is zero everywhere. Thus, we need to focus only on the kinetic energy.

From classical mechanics, the angular momentum $J = pr$ and the kinetic energy is given by

$$E = \frac{J^2}{2I} \qquad (32.1)$$

where I is the moment of inertia $I = mr^2$
From the de Broglie's relation $p = \frac{h}{\lambda}$

λ is the wavelength of the wavefunction ϕ for the particle on the ring
Since ϕ must be single-valued, ϕ must satisfy the cyclic boundary condition

$$\lambda = \frac{2\pi r}{m} \qquad m = 0, 1, 2, \ldots$$

($n=0 \Rightarrow \lambda \to \infty$ $\therefore$ ϕ has uniform amplitude)

$$J = pr = \frac{hr}{\lambda} = \frac{hmr}{2\pi r} = \frac{h}{2\pi} m = m\hbar \qquad m = 0, \pm 1, \pm 2, \ldots$$

$-$ $+$

Therefore $$E = \frac{J^2}{2I} = \frac{m^2\hbar^2}{2I} \qquad m = 0, 1, 2, \ldots \qquad (33.1)$$

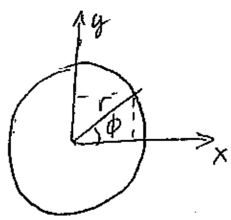
$\therefore$ Quantization of the rotational energy arises from the quantum mechanics requirement that the wavefunction ϕ must be single-valued.

Formal solution:

The Schrödinger equation for a particle on a ring in the x-y plane

$$-\frac{\hbar^2}{2m}\left(\frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial y^2}\right) = E\psi \qquad (33.2)$$

This is a second-order partial differential equation in both x and y. However, for the rotation of the particle on a ring, r is constant and only ϕ varies. Thus, it is more convenient to transform eq. (33.2) to polar coordinates (r, ϕ)



$$x = r\cos\phi$$
$$y = r\sin\phi$$

using the transformation (algebra is omitted here)

$$\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} = \frac{1}{r^2}\frac{d^2}{d\phi^2} \qquad (r \text{ is constant}) \qquad (33.3)$$

Eq. (33.2) becomes

$$-\frac{\hbar^2}{2mr^2}\frac{d^2\psi}{d\phi^2} = E\psi \qquad (33.4)$$

since $I = mr^2$

Eq. (33.4) $\Rightarrow$ $$\frac{d^2\psi}{d\phi^2} = -\frac{2IE}{\hbar^2}\psi \qquad (34.1)$$

(This is similar to the particle-in-a-box problem)

The general normalized solution is

$$\psi_m = \left(\frac{1}{2\pi}\right)^{1/2} e^{im\phi} \quad \text{where} \quad m = \pm\left(\frac{2IE}{\hbar^2}\right)^{1/2} \qquad (34.2)$$

Next, we impose the condition that ψ_m must be single-valued, or in other words, ψ_m must satisfy the cyclic boundary condition

$$\psi(\phi) = \psi(\phi + 2\pi)$$

or $$e^{im\phi} = e^{im(\phi+2\pi)} = \left(e^{i\pi}\right)^{2m} e^{im\phi}$$

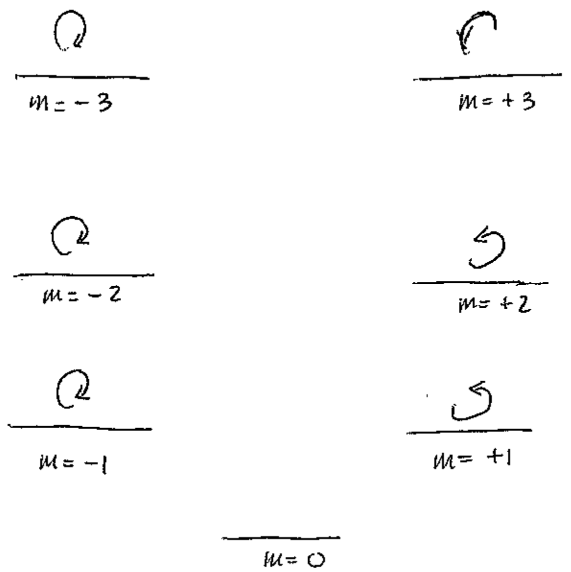
since $$e^{i\pi} = \cos\pi + i\sin\pi = -1$$

so $$e^{im\phi} = (-1)^{2m} e^{im\phi}$$

In order for this equality to be true, $2m$ must be an even number or m must be an integer

$$m = 0, \pm 1, \pm 2, \ldots$$

since $m = \pm\left(\frac{2IE}{\hbar^2}\right)^{1/2}$ $\Rightarrow$ $\boxed{E_m = \frac{m^2\hbar^2}{2I}}$ $\quad m = 0, \pm 1, \pm 2, \ldots$



Note for $m \neq 0$, the energy levels are <u>doubly degenerate</u> meaning we have two independent wavefunction for a given energy. This is expected since the energy of the particle moving on a ring would be independent of the direction (clockwise or counter-clockwise) of the particle motion

<u>An important fact about degeneracy</u>

Any linear combination of degenerate eigenfunctions is also an eigenfunction with the same eigenvalue.

$$\hat{H}\Psi_1 = E\Psi_1$$
$$\hat{H}\Psi_2 = E\Psi_2$$

consider $\Psi = c_1\Psi_1 + c_2\Psi_2$

$$\hat{H}\Psi = \hat{H}(c_1\Psi_1 + c_2\Psi_2) = c_1\hat{H}\Psi_1 + c_2\hat{H}\Psi_2$$
$$= c_1 E\Psi_1 + c_2 E\Psi_2 = E(c_1\Psi_1 + c_2\Psi_2)$$
$$\hat{H}\Psi = E\Psi$$

This property allow us to transform a set of degenerate wavefunctions to a more preferable but equivalent set without losing any generality.

For example, the solutions for the particle on a ring are

$$\psi_m = \frac{1}{\sqrt{2\pi}} e^{im\phi} \quad \text{for} \quad m = 0, \pm 1, \pm 2, \dots$$

for $m=0 \Rightarrow \phi_0 \quad \psi_0 = \frac{1}{\sqrt{2\pi}}$ is non-degenerate.

For $m \neq 0$, if we don't like to work with the complex number

$$e^{im\phi} = \cos(m\phi) + i\sin(m\phi)$$

we can take linear combinations

$$\frac{1}{\sqrt{2}}(\psi_m + \psi_{-m}) = \frac{1}{2\sqrt{\pi}}\left(\cos(m\phi) + \cancel{i\sin(m\phi)} + \cos(m\phi) - \cancel{i\sin(m\phi)}\right)$$

$$\frac{1}{i\sqrt{2}}(\psi_m - \psi_{-m}) = \frac{1}{i2\sqrt{\pi}}\left(\cancel{\cos(m\phi)} + i\sin(m\phi) - \cancel{\cos(m\phi)} + i\sin(m\phi)\right)$$

These factors keep the resulting wavefunction normalized.

$$\frac{1}{\sqrt{2}}(\psi_m + \psi_{-m}) = \frac{1}{\sqrt{\pi}}\cos(m\phi)$$

$$\frac{1}{i\sqrt{2}}(\psi_m - \psi_{-m}) = \frac{1}{\sqrt{\pi}}\sin(m\phi)$$

Thus,

$$\boxed{\begin{aligned} \psi_0(\phi) &= \frac{1}{\sqrt{2\pi}} \quad \text{for} \quad E_0 = 0 \\ \psi_m(\phi) &= \begin{cases} \frac{1}{\sqrt{\pi}}\cos(m\phi) \\ \frac{1}{\sqrt{\pi}}\sin(m\phi) \end{cases} \quad \text{for } m = 1, 2, \dots \quad \text{with } E_m = \frac{m^2\hbar^2}{2I} \end{aligned}}$$

are also solutions of the particle on a ring problem. We will use this property of degenerate wavefunctions later.

A more rigorous approach is to find the quantum mechanical operator corresponding to the angular momentum for the particle on a ring.

In classical mechanics, the angular momentum is

$$\vec{L} = \vec{r} \times \vec{p} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ x & y & z \\ p_x & p_y & p_z \end{vmatrix}$$

Thus, the z-component is

$$L_z = \hat{k} \begin{vmatrix} x & y \\ p_x & p_y \end{vmatrix} = (x p_y - y p_x)\, \hat{k}$$

we can transform L_z to the polar coordinate (r, ϕ) where r is constant.

$$\Rightarrow \quad \boxed{\hat{L}_z = \frac{\hbar}{i} \frac{\partial}{\partial \phi}} \quad \text{similar to} \quad \boxed{\hat{p}_x = \frac{\hbar}{i} \frac{\partial}{\partial x}}$$

$$\hat{L}_z \Psi_m(\phi) = \frac{\hbar}{i} \frac{1}{\sqrt{2\pi}} \frac{d}{d\phi} e^{im\phi} = m\hbar \frac{1}{\sqrt{2\pi}} e^{im\phi} = m\hbar \Psi_m(\phi)$$

$$\hat{L}_z \Psi_m = m\hbar \Psi_m$$

$\therefore$ $\Psi_m(\phi)$ are also eigenfunctions of the angular momentum operator of the particle on a ring. For this, $e^{im\phi}$ is also the prefered form of solution instead of

$$\left\{ \begin{array}{c} \cos(m\phi) \\ \sin(m\phi) \end{array} \right\}.$$

Furthermore, the increasing angular momentum is associated with the increasing number of nodes in the wavefunction.

Earlier, we mention that the particle on a ring is similar to the particle in a box problem, but the particle in a box has the zero-point energy whereas the particle on a ring has NOT.

Last time we involked the uncertainty principle to explain for the existence of the zero-point energy for the particle in a box. Again, we will use it to explain for the non-existence of the zero-point energy for the particle on a ring. To do this, we first find the probability for finding the particle between ϕ and $\phi + d\phi$

$$\Psi^*(\phi)\,\Psi(\phi)\,d\phi = \frac{1}{2\pi} e^{-im\phi} e^{im\phi} = \frac{1}{2\pi} d\phi$$

Since the probability for finding the particle anywhere is independent of ϕ, hence the location of the particle is completely indefinite. From the uncertainty principle, we can then precisely measure the angular momentum. Consequently, the minimum kinetic energy for the particle on a ring is zero.

Angular momentum

Last time, we used the de Broglie's relation to show the angular momentum for the particle on a ring is also quantized

$$J_z = m\hbar \qquad m = 0, \pm 1, \pm 2.$$

(z subscript indicates the rotation is about the z-axis)